DIN, DON, DAN - Geometry and Bells

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June 28, 2018

Din-Don-Dan – Bells and their Geometry

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Abstract. The design of bells has been a well-kept secret of bell founders over many centuries. Bells, religious and civil signal instruments, distinguish from other idiophone music instruments by their special sonorous sound due to a rich specter of overtones and, for the most, a long lasting reverberant sound. The shape of bells ranges from bicycle bells to the "classical" European church bells. Note that "bell" also can mean the end part of a wind instrument as e.g. a trumpet or a clarinet! Most bells are rotational symmetric, but there are also some with oval cross sections. Besides the classical" bell-shaped" design one finds spherical and conical bells, too. Some Asian bells are hit by a stick from outside, but most European church bells have a clapper, which strikes the bell from inside and form a double pendulum together with the swinging bell.
The meridian of a (rotationally symmetric) bell is called the "ribbon" of the bell, and its thickness and gradient is responsible for the sound specter of the bell. The size (and material) defines the fundamental note.
In mathematics there are many planar curves, which are called "bell curves", as e.g. the "Gaussian Bell Curve" and the "Versiera of M. Agnesi", c.f. [1]. At least affine versions of such curves might describe some bell ribbons very well. But it turns out that the bell-foundry Grassmayr (Innsbruck, Austria) uses circular bi-arcs for its bell-ribbon design, see [2] and [3].
The article does not deal with the physical properties of bells and their sound specter, which would need, besides deeper mathematical investigations, practical experiments.

Keywords: bells, bell curves, circular bi-arcs, double pendulum.

1 Introduction

In [4] the Chilean scientist Andres Gomberoff invites to discover the importance of sciences via connecting them with objects and processes in our common environment. He states correspondingly: “For many people science is not at all attractive, sometimes it is even terrifying. It is the same with zucchini: Some people don’t like it, but one cannot accuse the zucchini for that. It might be the matter of preparation! Similarly, we should rethink the presentation and teaching of science to improve its acception.”

This in mind, the paper considers bells as concrete objects familiar to everybody, and which allow to connect them with many concepts of geometry, mathematics, and physics. As basic objects for geometric investigations, bells can stimulate and motivate students to deeper study so well of historical facts and technical fabrication on one hand, and of geometry, mathematics and physics on the other. Even a fine art and cultural approach immediately comes to mind!
In the following we point to a geometric analysis of bells, thereby neglecting the physics of sound generation and specter. The latter would need practical experiments besides deeper mathematical investigation.

It seems interesting that “bell-shape” became an iconic term describing the form of flowers as well as a type of fashionable dresses, in spite the huge variety of bells, which are not at all bell-shaped. The end part of a wind instrument, too, is called its bell.
In the following we present some typical bells and focus finally on the classical rotational symmetric church bells.

2 Bells, their definition and standard forms

Bells belong to idiophonic music instruments mostly made of bronze or steel or even silver-tin alloy. The material’s elastic property is responsible for a long lasting reverberant sound. Bell instruments are fixed/hanged at their midpoint and hit at their rim, while gongs are hanged up at their rim and hit in the central region. Bells comprise singing bowls as well as bells for cows and sheep. Also bicycle bells and signal bells on hotel reception desks belong to bells even so they are not typically bell-shaped. Bells summoned slaves and butlers to their masters and called laborers home from field work. Bells call folks to church service. People still believe that the sound power density of rung bells influences thunderstorms and hailstorms in a relieving way. The uniform sound of bells was and is well suited for being used as religious and civil signal instruments.
Bells were known much before their appearance in Christianity, they occurred in the ancient empires of China, Babylon, Egypt, Greece, and Rome. Some almost bizarre forms of bells, where many of them were combined to large music instruments, were

excavated in China and, among other objects, show the mastership of Chinese founding. Shape and decoration of bells correspond to taste and style of the century they belong to. The knowledge, how to form the ribbon (meridian) of a bell to get a certain complex tone, was a rather well-kept secret by bell founder families through many generations, and it was/is based on practical experience than on mathematical/physical research.
The dynamics of a rung bell influenced and influences of course also the architecture of bell towers and installation of big bells often pounced on unexpected problems, as e.g. too small openings in the tower wall.

In the following we present some typical bell forms, starting with a classical church bell, see Fig.1. In Vienna (Austria) the biggest Austrian bell, the "Pummerin", heralds the New Year, see [5], [6]. The meridian of this bell reminds to a part of an (affinely distorted) Gaussian bell curve, c.f. e.g. [1]

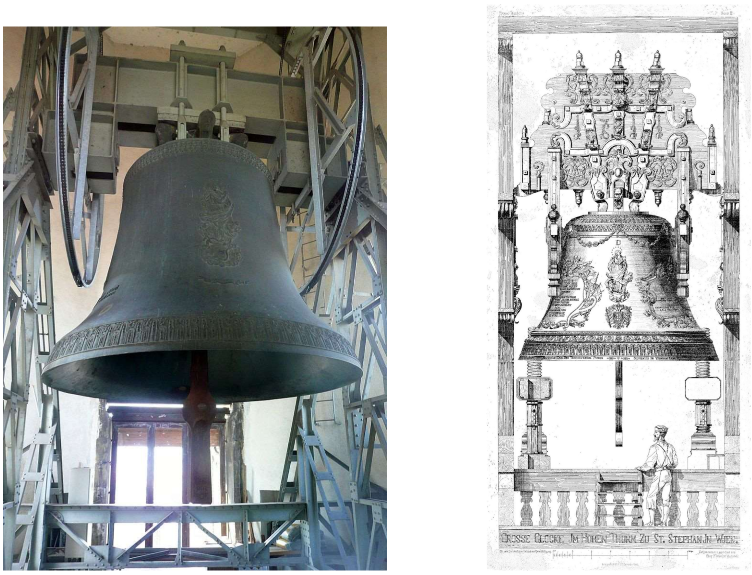

Fig. 1. The "Pummerin" (left), Austrian's largest and the world's fifth largest freely swinging bell. It has a 200 sec reverberance on hit tone c^0, 314 cm basis diameter and 294cm total height. It weighs 20130 kg, its the clapper weighs 630 kg. Cast in 1951 it succeeds a baroque "Pummerin" (right) with even deeper hit tone H^0, and which was destroyed in WW 2. Note the different positions of the axle-bearings. (Photos retrieved from [5].)

The next Figures present bells not having the classical bell-shape. Nonetheless their meridian can be associated with arcs of curves with simple mathematical descriptions. A few possibilities of such descriptions will be given in Chapter 4.

Fig. 2 (left) shows an African wooden cow bell with two clappers hitting the bell from outside. Fig. 2 (middle) depicts a sphere version of bells. (It relates to a worldwide known English Christmas song). Fig. 2 (right) presents an Austrian cow bell, a rural not rotational symmetric bell. As Fig. 3 (left) we present a set of simultaneously rung sacristy bells the shapes of which are spherical segments. Fig. 3 (right) shows an example of an altar bell, which is rung during service (Agnus Dei). It consists of four bells of different diameter and same height rung simultaneously. Truncated cones would be a very rough description of their form. A seemingly better description is proposed in chapter 4.

In Fig.4 we find a beehive shaped Asian bell. This form seems to be the oldest bell shape, dating back at least 4000 years. We find this form in Europe at bells of Romanesque style, Fig.5 (left and right).

On the way to modern bells we remind to baroque bells showing an extremal bell shape, Fig. 6 (left and right). In comparison, Fig. 7 (left and right) present modern bells.

Fig. 2. Left: African wooden cow-bell. Like a wooden xylophone, its “dry” sound has no reverberance. Its shape is roughly a truncated cone of revolution.

Middle: “… bells on bobtail ring …”: spherical bells at a horse harness. Like in a rattle, a freely movable stone in each bell causes the sound.

Right: An Austrian cow bell, a type of bells not being rotational symmetric. It is made of bronzed steel plate. It has a very characteristic sound with short reverberance. (All photos: G.W.)

Fig. 3. Left: Sacristy bells (Gurk, Austria). They are rung simultaneously by pulling one string, their shape relates to spherical segments. (Photo: G:W.) Right: Altar bells (Gurk, Austria). Their form represents an extremal stage of a bell's shape. (Photo: G.W.)

Fig. 4. Left and right: Chinese bell model with the typical Asian beehive shape. (Photo: G:W.)

Fig. 5. Left : “Lullus bell” founded in 1038, Bad Hersfeld, Germany. It shows the “original” beehive shape. (retrieved from [7]).
Right: Bell from Flatschach, Austria, founded in the 11th century. Its form shows a transition stage from beehive to classical bell shape. (Photo: G:W.)

Fig. 6. The baroque bells “Tullner Pummerin” (left) and “Zwölferin” (right), both from the parish church Tulln, Austria, both founded in 1752. (Retrieved from [8]).

Fig. 7. Left: modern bell “Peace” founded by Grassmayr in 1990, Gurk, Austria, Right: Shipsbell “Argus”, Cathedral of Bristol, Great Britain. (Both photos: G.W.)

We add some curiosities:
Bells made of non-metallic materials as glass, porcelain and wood, Fig.8. (A wooden bell is also that of Fig. 2, left). Fig. 9 shows hand bells of different cultures and materials. Fig. 10 presents an arrangement of signal bells of Dyrham castle, Great Britain. A complicated wire system made it possible to call a maiden or a servant from the basement up to manorial rooms.

Fig. 8. “Bells” consisting of non-metallic materials:
Left: Carillon of the “Zwinger”, Dresden, Germany, a set of bells consisting of Meißen porcelain. (retrieved from [9] and [10])
Middle: Glass bell, Danish Christmas decoration, showing an elongated bell shape.
Right: End piece of a clarinet, usually called its “bell”, (both photos: G.W.)

Fig. 9. European versus Asian bell shapes:
Left: a late baroque gilded table bell cast of bronce (Diocese Museum Gurk, Austria)
Middle: Asian hand bell, bronze.
Right: Pair of hand bells, made of silver-tin alloying, what causes a very long reverberation. (all photos: G.W.)

Fig. 10. Bell arrangement in Dyrham Castle, Great Britain. Each bell can be rung by pulling a wire in the manorial rooms of the castle. The bells are fixed at spiral springs, a pendulum enlarges ringing time. (Both photos: G.W.)

Finally, we point to bells, which are not surfaces of revolution. The most fascinating ones are the Japanese “Dotaku”, They mostly have an oval cross section, which makes it possible to evoke two different sounds by striking them at different points, Fig. 11. At the YouTube films [12] one can see and hear this effect. For the almost 3000 year old Chinese bell “zhong” Fig. 12 the sound difference is a major third, see [13] and [14].

Fig. 11. Left and middle**:** Japanese bells. Some of them have an elliptic cross section. (Retrieved from [11]., see also [12])

Fig. 12. Ritual bell „zhong“, Shaanxi, China, ~ 850–771 B.C, (Retrieved from [13]).

As another atypic bell we considericycle bells, Fig.13. They are rotational symmetric, the bell cup is fixed at its centre point, they are hit at their rim, so they fullfil the definition of a bell.

Fig. 13. Bicycle bells made of steel plate. The bell at left is rung from inside, the one at right from outside. (Photos: G.W.)

3 Bells, their parts and sounds

We describe the parts of a bell according to Grassmayr [2] from top to down, see Fig. 14 and [13]:

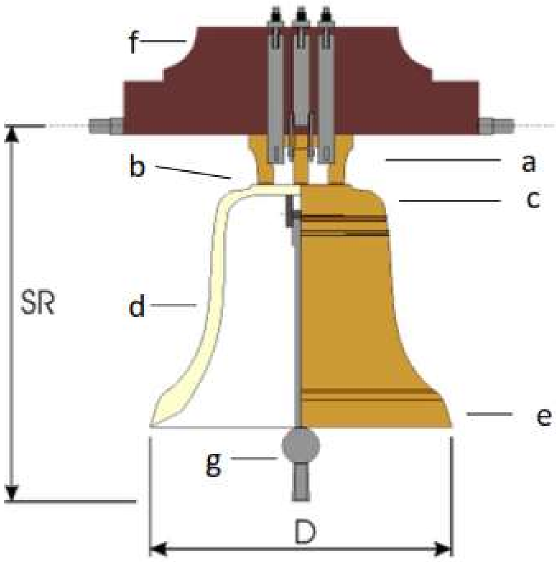

Fig. 14. Cross section of a modern bell. SR…swinging radius, D… sharpness diameter, a…g: see descriptions below.

a “Crown”: often hexagonal consisting of six bails in radial arrangement. Its purpose is to hang and fix the bell at the bearing construction (the “thrust bridge”) and the rotation axis. It has no influence to the bell’s sound.

b “Plate”: a more or less flat part as basis for the crown and the clapper mounting.

c “Shoulder”: the transition part from the plate to the bell shaped “ribbon”.

d “Ribbon”: the part responsible for the sound specter of the bell. The thickness of the ribbon increases downwards and is thickest there, where the clapper hits the bell.

e “Sharpness”: the end part of the bell, limited by two conical surfaces and ending a sharp edge.

f “Thrust bridge”: either wooden bearing construction, where the rotation axis is at the crown’s height, or a cranked steel construction, where the rotation axis is beneath the shoulder of the bell.

g “Clapper”: it is fixed at the plate bail by a loop. It consists of the “shaft” screwed to the loop, the drop-shaped “clapper bale”, which hits the bell, and the cylindrical “gudgeon pin”, an elongation of the bale to adjust the motion of the clapper.

Obviously, the clapper, together with the bell, forms a double pendulum. The swinging bell defines the excitation frequency for the clapper and the dimensions should be such, that the clapper hits the bell at its extreme positions. For bells with a cranked trust bridge the clapper’s motion is much more complicated. Bell founders try to receive a “falling clapper” instead a “flying clapper”. The two types of clapper hit the bell at opposite sides of the bell.

The fundamental tone of a bell mainly depends on the diameter of the sharpness, the thickness of the ribbon and the material. A thick sharpness, but rather thin ribbon gives a deeper tone as one with thicker ribbon. Steel bells of the same size sound a harmonic fifth higher than bronze bells. The raison for that is the higher sound velocity in steel than in bronze, c.f. [16]. (By the way, the velocity ratio is almost 3:2.)

Each bell is a unique copy. Its sound consists of a set of sine tones and a “hit sound”, which is a short lasting, metallic noise. Beside these tones one can hear an “undertone”, which cannot be measured directly and lies one octave below the deepest sine tone. It is generated by the first three to five sine tones, which are called “principal tones” and which have a relatively big wave amplitude. Its recognition is due to the physiology of the human ear. The other partial tones of the bell are called “mixture tones”..

Because of the bell-shape the sound specter, especially the set of mixture tones, is not that of a harmonic system of overtones. One might envision the vibration scheme of a bell as being analogous to the well-known Chladni figures on a circular disk, which is fixed at its center, see e.g. [17]. In [17] this Chladni like figure is described as a set of meridians and circles of latitude. The Japanese Dotaku (Fig. 11) show such a system of curves. It is not clear whether they are simply decoration or really have to do with the Chladni curve set of the Dotaku.

It should be mentioned that, for bells, one uses a gradation of an octave into 192 sub-tones. This means that a half tone interval is subdivided into 16 intervals. For example, $(e^1 - 2)$ would describe the main tone of a bell, an e^1 but $\frac{2}{16}$ of a half tone lower than a proper e^1.

The influence of the ribbon's thickness and gradient to the amplitudes of the mixture tones is still an open problem of physics and mathematics.

4 Geometric modelling of bell ribbons

A natural way of modelling a bell's ribbon would be to make a scan of it, extract its meridian and calculate a spline through meridian points. One could also ask for the best-fit conic section through the scanned meridian points. We would like to present another way, connecting the bell-shape with mathematical bell curves.

The first and most well-known curve is the "Gaussian bell curve". By an affine transformation (multiplying the ordinate values with a suitably chosen constant c) one can make the classical Gaussian bell curve better adapted to the shape of bells. We restrict ourselves to the equation

$$y = c.e^{-x^2}, c \geq 1 \tag{1}$$

Another typical and well-known bell curve is the "Versiera of M. Agnesi" with the standard equation

$$y = (x^2 + 1)^{-1} \tag{2}$$

Fig. 15 shows this curve (left) together with a modification of it (right)

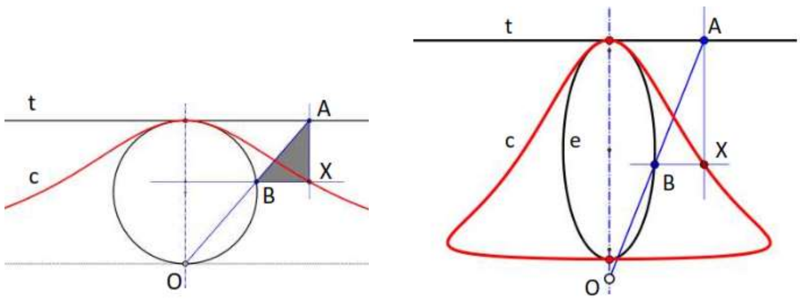

Fig. 15. The "Versiera of Maria Agnesi" (left) and a modification of it (right)

Of course, there are many functions, the graph or at least an arc of which shows a bell-shape. A simple example of such a function would be the sine-function, a more advanced one relates to a Fourier series. But simple constructions, too, can be used to derive bell-shaped curves. Fig. 16 shows a simple example of a bell curve construction. Some partial arcs of them suit very well to the beehive bells Fig. 4 and 5 and the baroque bells Fig. 6. The bell of a wind instrument (e.g. Fig. 8, right) can be formed either by rotating a parabolic arc or a piece of the graph of the function $y = e^x$, but also an arc of the Versiera, rotated around its asymptote would give a practicable solution.

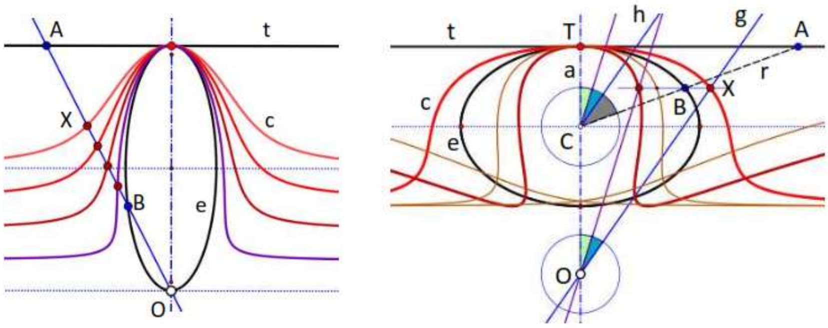

Fig. 16. Bell curve construction as intermediate curves between an ellipse and a line.
Left: The curves intersect the segments between A and B in constant ratio.
Right: The generating line g is parallel to h, with angle ratio $\sphericalangle ar : \sphericalangle ah = const.$

Finally, by generalizing (2), we get a set of curves with the equations

$$y = (x^{2\mathrm{p}} + 1)^{-1}, \ \ \mathrm{p} \in \mathbb{N}, \ \ (\mathrm{p} \in \mathbb{R}^+). \tag{3}$$

Those curves suit well to the bells shown in Fig. 3 and 13, see Fig. 17.

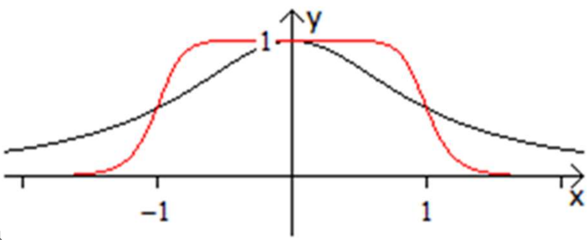

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Fig. 17. Curves with equations (3) for $p = 1$ (black, it is the Versiera) and $p = 5$ (red).

We end this chapter with the bell meridian construction of the Grassmayr foundry depicted in Fig. 18. The main part of so well the inner and the outer meridian of the ribbon is a “basket arch”, a circular bi-arc consisting of only two circular arcs. Each circular arc, rotated around the symmetry axis, generates a part of a torus, i.e. a surface of 4th degree. Rotating an arc of a conic section (e.g. an ellipse), such that it is a best-fitting replacement of the basket arch, we again would receive a surface of 4th degree.

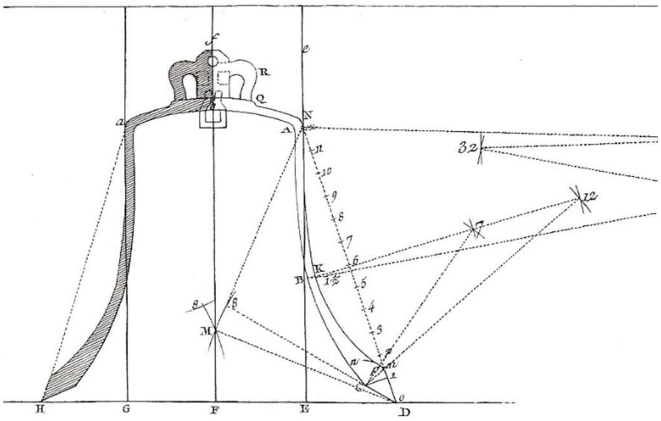

Fig. 18. Bell meridian construction as circular bi-arc used by the Grassmayr foundry. The numbers at the circle centers give the radii in parts of the segment from D to N (Courtesy Grassmayr P. and J., Austria).

5 Phenomena occurring when ringing the bell

In this concluding chapter we point to phenomena of bells without going into the details.

a) We start with mentioning that the bell, together with its clapper, forms a double pendulum, see Fig. 19.

The swinging bell actuates a motion of the clapper and their mass dimensions have to be such that the clapper motion resonates with that of the bell. Thereby the clapper's double swing stops at or at least near the slack point position of the bell. At the moment of contact with the bell the kinetic energy of the clapper causes the vibrations of the bell and thus its sound. A too weak collision causes a jingling sound, a too hard kick can damage the bell and the clapper. Note that a bell resp. its clapper might weigh 20t resp. up to 1t. In addition, the two motions of bell and clapper should be such that the time of contact is as short as possible, otherwise the bell's vibration will be damped and the sound not clear.

b) The influence of the trust bridge to the bell's swinging: the heavier the crossbeam over the rotation axis, and the further its gravity center from the axis, the slower is the pendulum motion of the bell.

The same effect can be seen at bells mounted to cranked trust bridges. The hinge for the clapper then must be either over or under the swing axis of the bell to be forced to an induced motion by the rung bell. In case it is above the main axis, the induced motion of the clapper is opposite to that of the bell and the clapper hits the bell at the downwards motion. It is called "falling clapper" in contrast to the one shown in Fig. 19, which is named "flying clapper".

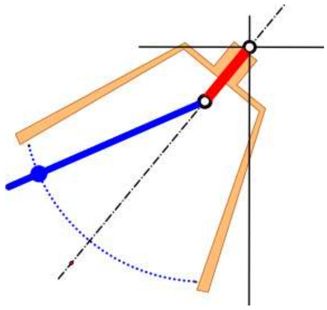

Fig. 19. Schematic sketch of a bell as a double pendulum. The swing axis of the bell is above the crown, the bell has therefore a "flying clapper".

c) Doppler effect: For a listener positioned in the plane of the double swing of a big bell, the bell's for and back motion causes already a slight but noticeable undulation of the sound.

d) Dynamics: The maximum deflection of a large bell must be limited, as it is responsible of its potential energy at the slack point positions to avoid excessive load of the axle bearing. This load is highest, when the bell passes through the low point of its path.

This dynamic effect was already known to Leonardo da Vinci, who proposed an axel bearing consisting of segments of three large wheels, see [18, p.172] and the schematic sketch Fig. 20. This idea allows bigger diameter for the axel shaft. The rotation angle of this axel is transmitted to the three bearing sectors and diminished according to the ratio of axel and sectors.

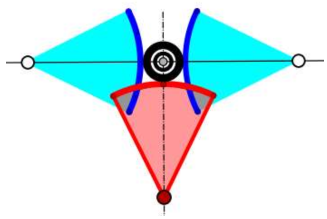

Fig. 20. Leonardo da Vinci's proposal for an axel bearing of a bell's trust bridge.

Final remark: Even so the presented topic seems to be rather special, it reveals many different approaches for investigation. With this material we want to suggest students and teachers to look closer to a topic, where one can discover and train concepts of geometry and mathematics or just to plunge into a part of human culture. Internet and references provide lots of information, but also tells about open problems.

Acknowledgement

The authors thank the Bell Foundry Grassmayr for information about actual bell founding and Prof. Dr. Kenjiro Suzuki for information about the Japanese Dotaku.

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