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Soliton Solution of The Hirota Equation by Modified Analytical Method

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Abstract. Nonlinear Schrodinger equation plays significant role in optical soliton communication. To improve transmission speed in optical soliton communications, high-power and ultra-short optical pulses should be used. And generalized third order nonlinear Schrödinger equation is used in optical fibers for describing ultra-short pulses. In this study, we utilized generalized Riccati mapping method to get numerous kinds of optical solitons of Hirota equation which is special generalized nonlinear Schrödinger equation of third order.

Keywords: Hirota equation, Generalized Riccati mapping method, Optical soliton

1 Introduction

Solitary waves, which entered the scientific literature with the studies of John Scott Russell in 1834, were not paid much attention until the 1960s. The studies in the Los Alamos National Laboratory of Norman J. Zabusky and Martin David Kruskal increased the importance of solitary waves. Kruskal and Zabusky show that the Fermi-Pasta-Ulam-Tsingou model associated with the propagation of equal mass-longitudinal waves by certain nonlinear springs in a one-dimensional lattice pairs are modeled by the KdV equation. Kruskal and Zabusky discovered that solitary waves would emerge from collision having the same shape and speeds which they entered when experimenting with the numerical solution of the KdV equation. They called these waves "solitons" which show particle behavior like photon, proton. etc. In Greek, the suffix "on" means solitary. Mathematicians often prefer to distinguish between solitons and solitary waves. They call solitons solitary waves that do not deform after colliding with other solitons. In this case, solitary waves are of a wider variety than solitons.The next question is how are solitons formed? Solitons arise from the balance between the nonlinear term and the nonlinear term causing the dispersion in partial differential equation. Does every partial differential equation form a soliton? KdV equation, Boussinesq equation, Burgers equation, Sine-Gordon equation and Schrödinger equation are the most famous partial differential equations that cause to soliton solutions[1, 2].

2 Governing Model

In this paper, our governing equation is a generalized third order Schrödinger equation reads as

$$iU_t + aU_{xx} + bU|U|^2 + icU_{xxx} + ih|U|^2 U_x = 0, \tag{1}$$

where x is propagation variable, t is time variable and a, b, c, h are real constants. This equation is known as Hirota-type generalized nonlinear Schrödinger equation. For the values $c = h = 0$, it is reduced to the Schrödinger equation and for the values $a = b = 0$, the equation reduces to the modified KdV equation [3, 6].

3 The Generalized Riccati Mapping Scheme

We give algorithm of the Generalized Riccati mapping method for seeking soliton solution of nonlinear partial differential equations. Suppose general form of a nonlinear PDE in two independent as

$$P\left(U, U_x, U_t, U_{xx}, ...\right) = 0, \tag{2}$$

where the $U(x,t)$ is unknown function and P polynomial function is including with linear and nonlinear terms of the derivatives of the $U(x,t)$. The basic steps of this technique are given below:

Step 1: We suppose complex the traveling wave transform independent variables into the single variables as

$$U\left(x,t\right) = \chi\left(\xi\right) e^{iR}, \chi\left(\xi\right) = \sum_{j=-N}^{N} A_j \varphi^j\left(\xi\right) \tag{3}$$

where $R = \delta x + \lambda t + \theta$ and $\xi = kx + \omega t$. χ and R are amplitude of wave profile with the parameters δ, λ,and θ. ω is frequency and k is wave length. A_j are real constants and $\varphi(\xi)$ satisfies the below equation

$$\varphi'\left(\xi\right) = p + q\varphi\left(\xi\right) + r\varphi^2\left(\xi\right) \tag{4}$$

where p, q, r are real constants.

Step 2: Integer N is get by the balancing higher order nonlinear and linear terms on Eq. (3) and set of the coefficients A_j, k, ω, p, q and r are can be obtained.

Step 3: Inserting Eq. (3) with Eq. (4) into Eq.(2), choosing the coefficients of dissimilar exponents of $\varphi^j(\xi)$ to zero, get a set of equations, which is solved by using software Mathematica, then the parameter values can be obtained.

Step 4: If we place these values in Eq. (3), then Eq. (2) has a solution [7, 10].

4 Application of the method to Hirota NLSE

Since Eq. (1) is complex, when we inserting Eq. (3) into Eq. (1), we get real and imaginary parts, respectively as

$$\left(-\lambda - a\delta^2 + c\delta^3\right)\chi + (b - h\delta)\chi^3 + \left(ak^2 - 3ck^2\delta\right)\chi'' = 0, \tag{5}$$

$$\left(\omega + 2ak\delta - 3ck\delta^2\right)\chi' + hk\chi^2\chi' + ck^3\chi^{(3)} = 0. \tag{6}$$

Integrating Eq. (6) once and taking integration constants zero, we have as

$$(\omega + 2ak\delta - 3ck\delta^2)\chi + \frac{hk}{3}\chi^3 + ck^3\chi'' = 0. \tag{7}$$

As it is seen Eq. (5) and (7) are similar. Thus, we get the following relations between the coefficients

$$a = c\,(k + 3\delta)\,, h = \frac{3b}{k + 3\delta}, \lambda = -2ck^2\delta - 4ck\delta^2 - 2c\delta^3 - \omega, \tag{8}$$

By using homogenous balancing principle on Eq. (5), we get the solution as

$$\chi\,(x,t) = \frac{A_{-1}}{\varphi} + A_1\varphi + A_0 \tag{9}$$

Substituting Eq. (9) and (4) into Eq. (7) and collecting the same powers of to zero, our system of equations are consisting in parameters $A_{-1}, A_0, A_1, k, \delta, \omega$, a, c, h, p, q, r. Solving these system of equation with the aid of Mathematica, we attain following cases: Case 1:

$$A_{-1} = 0, A_0 = \frac{i\sqrt{\frac{3c}{2}}kq}{\sqrt{h}}, A_1 = \frac{i\sqrt{6c}kr}{\sqrt{h}}, \omega = k\left(3c\delta^2 - 2a\delta\right) + \frac{ck^3}{2}\left(q^2 - 4pr\right). \tag{10}$$

Case 2:

$$A_{-1} = \frac{i\sqrt{6c}kp}{\sqrt{h}}, A_0 = \frac{i\sqrt{\frac{3}{2}}ckq}{\sqrt{h}}, A_1 = 0, \omega = k\left(3c\delta^2 - 2a\delta\right) + \frac{ck^3}{2}\left(q^2 - 4pr\right). \tag{11}$$

Case 3:

$$A_{-1} = \frac{i\sqrt{6c}kp}{\sqrt{h}}, A_0 = 0, A_1 = \frac{i\sqrt{6c}kr}{\sqrt{h}}, \omega = k\left(4ck^2pr - 2a\delta + 3c\delta^2\right). \tag{12}$$

For Case 1, the wave solutions of Eq. (1) in different soliton types are constructed in the following types:

Set I: When $q^2 - 4p > 0$and $qr \neq 0 (pr \neq 0)$,

$$U_1\,(\xi) = -\frac{k\sqrt{3c\,(q^2 - 4pr)}\tanh\left(\frac{1}{2}\sqrt{q^2 - 4pr}\xi\right)}{\sqrt{2h}}e^{i\left(\delta x + \lambda t + \theta + \frac{\pi}{2}\right)}, \tag{13}$$

$$U_2(\xi) = -\frac{k\sqrt{3c\left(q^2-4pr\right)}\coth\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}{\sqrt{2h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (14)$$

$$U_3(\xi) = -\frac{k\sqrt{3c(q^2-4pr)}sech\left(\sqrt{q^2-4pr}\xi\right)\left(\sinh\left(\sqrt{q^2-4pr}\xi\right)+i\right)}{\sqrt{2h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (15)$$

$$U_4(\xi) = -\frac{k\sqrt{3c(q^2-4pr)}csch\left(\sqrt{q^2-4pr}\xi\right)\left(\cosh\sqrt{q^2-4pr}\xi\pm i\right)}{\sqrt{2h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (16)$$

$$U_5(\xi) = -\frac{k\sqrt{3c(q^2-4pr)}\left(1+\tanh\left(\frac{1}{4}\sqrt{q^2-4pr}\xi\right)\pm\coth\frac{1}{4}\sqrt{q^2-4pr}\xi\right)}{2\sqrt{2h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (17)$$

$$U_6(\xi) = \frac{\sqrt{3c}k\left(\sqrt{(A^2+B^2)(q^2-4pr)}-A\sqrt{q^2-4pr}\cosh\left(\sqrt{q^2-4pr}\xi\right)\right)}{\sqrt{2h}\left(A\sinh\left(\sqrt{q^2-4pr}\xi\right)+B\right)} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (18)$$

$$U_7(\xi) = -\frac{\sqrt{3c}k\left(\sqrt{(B^2-A^2)(q^2-4pr)}+A\sqrt{q^2-4pr}\cosh\left(\sqrt{q^2-4pr}\xi\right)\right)}{\sqrt{2h}\left(A\sinh\left(\sqrt{q^2-4pr}\xi\right)+B\right)} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (19)$$

where $A \neq 0$ and$B \neq 0$ are two reals and $B^2 - A^2 > 0$,

$$U_8(\xi) = \frac{\sqrt{\frac{3c}{2}}k\left(\frac{4pr\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}{\sqrt{q^2-4pr}\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)-q\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}+q\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (20)$$

$$U_9(\xi) = \frac{\sqrt{\frac{3c}{2}}k\left(\frac{4pr\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}{\sqrt{q^2-4pr}\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)-q\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}+q\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (21)$$

$$U_{10}(\xi) = \frac{\sqrt{\frac{3c}{2}}k\left(q\pm\frac{4pr\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}{\mp q\cosh\left(\sqrt{q^2-4pr}\xi\right)+\sqrt{q^2-4pr}\left(i\pm\sinh\left(\sqrt{q^2-4pr}\xi\right)\right)}\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \qquad (22)$$

$$U_{11}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q+\frac{4pr\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}{\sqrt{q^2-4pr}\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)-q\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)}\right)}{\sqrt{h}}$$

$$e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{23}$$

$$U_{12}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q\sqrt{q^2-4pr}\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)-\left(q^2-4pr\right)\sinh\left(\frac{1}{4}\sqrt{q^2-4pr}\xi\right)\right)}{\sqrt{h}\left(\sqrt{q^2-4pr}\cosh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)-q\sinh\left(\frac{1}{2}\sqrt{q^2-4pr}\xi\right)\right)}$$

$$e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}. \tag{24}$$

Set II: When $q^2-4p<0$and $qr\neq 0(pr\neq 0)$,

$$U_{13}(\xi)=\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\tan\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{25}$$

$$U_{14}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\cot\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{26}$$

$$U_{15}(\xi)=\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\left(\pm\sec\left(\sqrt{4pr-q^2}\xi\right)+\tan\left(\sqrt{4pr-q^2}\xi\right)\right)}{\sqrt{h}}$$

$$e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{27}$$

$$U_{16a}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\cot\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)},$$

$$U_{16b}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq^2}{2}}\tan\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{h}}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{28}$$

$$U_{17}(\xi)=-\frac{k\sqrt{6cpr-\frac{3cq}{2}^2}\left(\coth\left(\frac{1}{4}\sqrt{4pr-q^2}\xi\right)-\tan\left(\frac{1}{4}\sqrt{4pr-q^2}\xi\right)\right)}{2\sqrt{h}}$$

$$e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{29}$$

$$U_{18}(\xi)=\pm\frac{\sqrt{\frac{3c}{2}}k\left(\sqrt{(A^2-B^2)(q^2-4pr)}\mp A\sqrt{4pr-q^2}\cos\left(\sqrt{4pr-q^2}\xi\right)\right)}{\sqrt{h}\left(A\sinh\left(\sqrt{4pr-q^2}(\xi)\right)+B\right)}$$

$$e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{30}$$

$$U_{19}(\xi)=-\frac{\sqrt{\frac{3c}{2}}k\left(\sqrt{(A^2-B^2)(q^2-4pr)}\pm A\sqrt{4pr-q^2}\cos\left(\sqrt{4pr-q^2}\xi\right)\right)}{\sqrt{h}\left(A\sin\left(\sqrt{4pr-q^2}\xi\right)+B\right)}$$

$$e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \tag{31}$$

where $A\neq 0$ and$B\neq 0$ are two reals and $A^2-B^2>0$,

$$U_{20}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q+2qr-\frac{4pr\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{4pr-q^2}\sin\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)+q\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (32)$$

$$U_{21}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q+2qr+\frac{4pr\sin\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{4pr-q^2}\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)-q\sin\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (33)$$

$$U_{22}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q-\frac{4pr\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{4pr-q^2}\left(\pm1+\sin\left(\sqrt{4pr-q^2}\xi\right)\right)+q\cos\left(\sqrt{4pr-q^2}\xi\right)}\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (34)$$

$$U_{23}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q\pm\frac{4pr\sin\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}{\sqrt{4pr-q^2}-q\sin\left(\sqrt{4pr-q^2}\xi\right)+\sqrt{4pr-q^2}\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)}\right)}{\sqrt{h}} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (35)$$

$$U_{24}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(q\sqrt{4pr-q^2}\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)+\left(4pr-q^2\right)\sin\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)\right)}{\sqrt{h}\left(\sqrt{4pr-q^2}\cos\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)-q\sin\left(\frac{1}{2}\sqrt{4pr-q^2}\xi\right)\right)} e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}. \quad (36)$$

Set III: When $p=0$ and $qr\neq 0$,

$$U_{25}(\xi)=\frac{\sqrt{\frac{3c}{2}}kq\left(1-de^{q\xi}\right)}{\sqrt{h}\left(1+de^{q\xi}\right)}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (37)$$

$$U_{26}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(-2rq\left(e^{q\xi}\right)+qr\left(d+e^{q\xi}\right)\right)}{\sqrt{h}r\left(d+e^{q\xi}\right)}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}, \quad (38)$$

where d is an arbitrary constant.

$$U_{27}(\xi)=\frac{\sqrt{\frac{3c}{2}}k\left(c_1q-2r+q^2\xi\right)pe^{p\xi}}{\sqrt{h}\left(c_1+q\xi\right)}e^{i\left(\delta x+\lambda t+\theta+\frac{\pi}{2}\right)}. \quad (39)$$

where c_1 is an arbitrary constant.

Similarly, further results can be obtained from other cases in a more generalized form.

Conclusion

In this study, the Generalized Riccati mapping method has been successfully applied to find different types of soliton, solitary waves, trigonometric function and other solutions of generalized third order Hirota equation. From this perspective, we can say that the method we use is effective, precise, brief and applicable to different nonlinear partial differential equations.

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