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Weight Distributions of a Class of Linear Codes

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***Abstract*—Linear codes with a few weights have many applications in secret sharing schemes, authentication codes, association schemes and strongly regular graphs, and they are also of importance in consumer electronics, communications and data storage systems. In this literature, based on the theory of defining sets, we present a class of five-weight linear codes over $\mathbb{F}_p$(p is an odd prime). Then, we use exponential sums to determine its weight distributions.**

***Index Terms*—linear codes, Gauss sums, weight distributions.**

I. Introduction

Let p be an odd prime and m be a positive integer. Let $\mathbb{F}_q$ denote the finite field with $q = p^m$ elements and $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$. Let Tr and Tr_s denote the absolute trace function [12] from $\mathbb{F}_q$ onto $\mathbb{F}_p$ and $\mathbb{F}_{p^s}$ onto $\mathbb{F}_p$, respectively.

An $[n, k, d]$ linear code $\mathcal{C}$ over $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming distance d. Let A_i be the number of codewords with Hamming weight i in a code $\mathcal{C}$. The weight enumerator of $\mathcal{C}$ is defined by

$$1 + A_1 z + A_2 z^2 + \ldots + A_n z^n,$$

and the sequence $(1, A_1, \ldots, A_n)$ is called the weight distribution of $\mathcal{C}$ [8]. If the number of the nonzero $A_i (1 \leq i \leq n)$ is t, then we say $\mathcal{C}$ a t-weight code.

The weight distributions are an important research object in coding theory, because it contains the useful information that is helpful to estimate the error correcting capability. It is of significance in combination schemes [2], authentication codes [5], combination designs [3], and secret sharing schemes [15]. In 1972, a new construction of linear codes was introduced firstly in references. In 2007, professor Ding and Niederreiter presented that linear codes with a few weights can be constructed by selecting properly the defining set, which is a subset of finite fields. Namely, let $D = \{d_1, d_2, \ldots, d_n\} \subseteq F_q$. The linear code $\mathcal{C}_D$ of length n over $\mathbb{F}_p$ is defined by

$$\mathcal{C}_D = \{(Tr(xd_1), Tr(xd_2), \ldots, Tr(xd_n)) : x \in \mathbb{F}_q\}, \tag{1}$$

here the set D is called the defining set of linear code $\mathcal{C}_D$. From then on, many works have been done in this study by scholars (See Refs. [1], [6], [7], [9], [10], [11], [13], [14], [16] for example). In particular, there are optimal codes and almost optimal codes in the literature [7], [10], [11], [13], [14], [16].

In this literature, with the construction method as mentioned above, we obtain a class of linear codes with a few weights. Meanwhile, we determine their weight distributions by character sums over finite filed.

II. Preliminaries

An additive character of $\mathbb{F}_q$ is a non-zero function χ from $\mathbb{F}_q$ to the set of complex numbers of absolute value 1 such that $\chi(x+y) = \chi(x)\chi(y)$ for any pairs $(x, y) \in \mathbb{F}_q^2$. For each $u \in \mathbb{F}_q$, the function

$$\chi_u(v) = \zeta^{Tr(uv)}, \ v \in \mathbb{F}_q$$

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TABLE I
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_D$

Weight	Multiplicity
0	1
$p^{2m-2}+p^{m+s-2}$	$p-1$
$(p-1)(p^{2m-3}+p^{m+s-3})$	$p^m(p^m-p)$
$p^{2m-3}(p-1)$	$p^{s-1}-\frac{3}{2}p^s+\frac{1}{2}p^{s+1}+p^{m-1}-\frac{1}{2}p^m+\frac{1}{2}p^{m+1}-1$
$p^{2m-3}(p-1)+2p^{m+s-2}$	$p^{s-1}-\frac{3}{2}p^s+\frac{1}{2}p^{s+1}+p^{m-1}-\frac{3}{2}p^m+\frac{1}{2}p^{m+1}$
$p^{2m-3}(p-1)+p^{m+s-2}$	$-2p^{s-1}-p+1+3p^s-p^{s+1}-2p^{m-1}+2p^m$

denotes an additive character of $\mathbb{F}_q$, where $\zeta = e^{2\pi i/p}$ is a primitive p-th root of unity and $i=\sqrt{-1}$. Since $\chi_0(v)=1$ for all $v\in\mathbb{F}_q$, we call it the trivial additive character of $\mathbb{F}_q$. We call χ_1 the canonical additive character of $\mathbb{F}_q$ and $\chi_u(x)=\chi_1(ux)$ for all $u\in\mathbb{F}_q$. The additive character satisfies the orthogonal property [12], that is

$$\sum_{v\in\mathbb{F}_q}\chi_u(v)=\begin{cases} q, & \text{if } u=0,\\ 0, & \text{if } u\neq 0.\end{cases}$$

Let g be a fixed primitive element of $\mathbb{F}_q$. For each $j=0,1,\ldots,q-2$, the function $\lambda_j(g^k)=e^{2\pi ijk/(q-1)}$ for $k=0,1,\ldots,q-2$ defines a multiplicative character of $\mathbb{F}_q$, we extend these characters by setting $\lambda_j(0)=0$. It satisfies

$$\sum_{v\in\mathbb{F}_q^*}\lambda_j(v)=\begin{cases} q-1, & \text{if } j=0,\\ 0, & \text{if } j\neq 0.\end{cases}$$

The following text we refer to $\eta=\lambda_{(q-1)/2}$ and $\eta'=\lambda_{(p-1)/2}$ the quadratic characters over $\mathbb{F}_q$ and $\mathbb{F}_p$, respectively. The Gauss sums over $\mathbb{F}_q$ and $\mathbb{F}_p$ are defined respectively by

$$G(\lambda)=\sum_{c\in\mathbb{F}_q^*}\lambda(v)\chi_1(v)$$

and

$$G'(\lambda')=\sum_{c\in\mathbb{F}_p^*}\lambda'(v)\chi_1'(v),$$

where λ' and χ_1' are the canonical multiplicative and additive characters of $\mathbb{F}_p$, respectively.

III. MAIN RESULTS

In this section, we present the main results, including the construction, the parameters and the weight distributions of the linear codes $\mathcal{C}_D$. The proofs will be given in section IV.

We firstly select the defining set

$$D=\{(x_1,x_2)\in\mathbb{F}_q^2 : Tr(x_1^{p^s+1})=1, Tr(x_2)=1\} \quad (2)$$

to construct linear code

$$\mathcal{C}_D=\{(Tr(ax_1+bx_2)_{(x_1,x_2)\in D}) : a,b\in\mathbb{F}_q\}, \quad (3)$$

where $m=2s$ and s is a positive integer.

Define

$$Sp=\{x^2 : x\in\mathbb{F}_p^*\} \text{ and } NSp=\mathbb{F}_p^*\backslash Sp. \quad (4)$$

Theorem 1: With the notation above, the weight distributions of the code $\mathcal{C}_D$ with the parameter $[p^{2m-2}+p^{m+s-2}, 2m]$ is shown in Table I. Obviously, the code is 5-weight.

Example 1: If $(p,m)=(5,2)$, then the code $\mathcal{C}_D$ has parameters $[30,4,20]$ and weight enumerator $1+60z^{20}+500z^{24}+24z^{25}+40z^{30}$, which from Magma program confirmed the result in Theorem 1.

Example 2: If $(p,m)=(3,2)$, then the code $\mathcal{C}_D$ has parameters $[12,4,6]$ and weight enumerator $1+12z^6+54z^8+8z^9+6z^{12}$, which from Magma program confirmed the result in Theorem 1.

Example 3: If $(p,m)=(3,4)$, then the code $\mathcal{C}_D$ has parameters $[810,8,486]$ and weight enumerator $1+110z^{486}+6318z^{540}+100z^{567}+30z^{648}+2z^{810}$, which from Magma program confirmed the result in Theorem 1.

IV. AUXILIARY LEMMAS AND PROOFS OF THE MAIN RESULTS

We begin this section by introducing some known facts on exponential sums.

Lemma 1: [4] Let $m=2s$, where s is a positive integer. For any $a\in\mathbb{F}_{p^s}^*$ and $b\in\mathbb{F}_{p^m}$, we have

$$\sum_{x\in\mathbb{F}_q}\zeta^{Tr_s(ax^{p^s+1})+Tr(bx)}=-p^s\zeta^{-Tr_s(\frac{b^{p^s+1}}{a})}.$$

Lemma 2: [12] If $f(x)=a_2x^2+a_1x+a_0\in\mathbb{F}_q[x]$, where $a_2\neq 0$, then

$$\sum_{x\in\mathbb{F}_q}\zeta^{Tr(f(x))}=\zeta^{Tr(a_0-a_1^2(4a_2)^{-1})}\eta(a_2)G(\eta).$$

Lemma 3: [12] With the notation above, we have

$$G(\eta)=(-1)^{m-1}\sqrt{(p^*)^m}, G'(\eta')=\sqrt{p^*},$$

where $p^* = (-1)^{\frac{p-1}{2}} p$.

The following Lemmas 4-5 are essential to compute the length and the weight distributions of linear code $\mathcal{C}_D$.

Lemma 4: The length of the $\mathcal{C}_D$ is

$$\begin{aligned} n &= |\{(x_1, x_2) \in \mathbb{F}_q^2 : Tr(x_1^{p^s+1}) = 1, Tr(x_2) \\ &\quad = 1\}| = p^{2m-2} + p^{m+s-2}. \end{aligned} \tag{5}$$

Proof: Since

$$\begin{aligned} n &= |\{(x_1, x_2) \in \mathbb{F}_q^2 : Tr(x_2) = 1, Tr(x_1^{p^s+1}) \\ &\quad = 1\}| \\ &= \sum_{x_1, x_2 \in \mathbb{F}_q} (\frac{1}{p} \sum_{y_1 \in \mathbb{F}_p} \zeta^{y_1(Tr(x_1^{p^s+1})-1)}) \\ &\quad \cdot (\frac{1}{p} \sum_{y_2 \in \mathbb{F}_p} \zeta^{y_2(Tr(x_2)-1)}) \\ &= p^{2m-2} + \frac{1}{p^2}(\Omega_1 + \Omega_2 + \Omega_3), \end{aligned}$$

where

$$\begin{aligned} \Omega_1 &= \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{y_2 \in \mathbb{F}_p^*} \zeta^{y_2 Tr(x_2) - y_2} \\ &= \sum_{y_2 \in \mathbb{F}_p^*} \zeta^{-y_2} \sum_{x_1, x_2 \in \mathbb{F}_q} \zeta^{y_2 Tr(x_2)} \\ &= 0, \\ \Omega_2 &= \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{y_1 \in \mathbb{F}_p^*} \zeta^{y_1 Tr(x_1^{p^s+1}) - y_1} \\ &= \sum_{y_1 \in \mathbb{F}_p^*} \zeta^{-y_1} \sum_{x_1, x_2 \in \mathbb{F}_q} \zeta^{y_1 Tr(x_1^{p^s+1})} \\ &= p^{m+s}, \\ \Omega_3 &= \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{y_1 \in \mathbb{F}_p^*} \zeta^{y_1 Tr(x_1^{p^s+1}) - y_1} \\ &\quad \cdot \sum_{y_2 \in \mathbb{F}_p^*} \zeta^{y_2 Tr(x_2) - y_2} \\ &= \sum_{y_1 \in \mathbb{F}_p^*} \zeta^{-y_1} \sum_{y_2 \in \mathbb{F}_p^*} \zeta^{-y_2} \sum_{x_1 \in \mathbb{F}_q} \zeta^{y_1 Tr(x_1^{p^s+1})} \\ &\quad \cdot \sum_{x_2 \in \mathbb{F}_q} \zeta^{y_2 Tr(x_2)} \\ &= 0. \end{aligned}$$

Then $n = p^{2m-2} + p^{m+s-2}$. □

Let

$$\begin{aligned} T &= |\{(x_1, x_2) \in \mathbb{F}_q^2 : Tr(x_1^{p^s+1}) = 1, Tr(x_2) \\ &\quad = 1, Tr(ax_1 + bx_2) = 0\}|. \end{aligned}$$

For any $a, b \in \mathbb{F}_q$ and $\mathbf{c}(a, b) \in \mathcal{C}_D$, the weight of $\mathbf{c}(a, b)$ is as follows

$$w(\mathbf{c}(a, b)) = n - T. \tag{6}$$

Then

$$\begin{aligned} T &= \sum_{x_1, x_2 \in \mathbb{F}_q} (\frac{1}{p} \sum_{z_1 \in \mathbb{F}_p} \zeta^{z_1 Tr(x_1^{p^s+1}) - z_1})(\frac{1}{p} \qquad (7) \\ &\quad \cdot \sum_{z_2 \in \mathbb{F}_p} \zeta^{z_2 Tr(x_2) - z_2})(\frac{1}{p} \sum_{z_3 \in \mathbb{F}_p} \zeta^{z_3 Tr(ax_1 + bx_2)}) \\ &= \frac{n}{p} + \frac{1}{p^3}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4), \qquad (8) \end{aligned}$$

where φ_1, φ_2 and φ_3 are showed as follows, and φ_4 is showed at the top of next page.

$$\begin{aligned} \varphi_1 &= \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{z_3 \in \mathbb{F}_p^*} \zeta^{z_3 Tr(ax_1 + bx_2)} \\ &= \sum_{z_3 \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q} \zeta^{z_3 Tr(ax_1)} \sum_{x_2 \in \mathbb{F}_q} \zeta^{z_3 Tr(bx_2)} \\ &= \begin{cases} p^{2m}(p-1), & a = b = 0, \\ 0, & otherwise, \end{cases} \end{aligned}$$

$$\begin{aligned} \varphi_2 &= \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{z_2 \in \mathbb{F}_p^*} \zeta^{z_2(Tr(x_2)-1)} \\ &\quad \cdot \sum_{z_3 \in \mathbb{F}_p^*} \zeta^{z_3 Tr(ax_1 + bx_2)} \\ &= \sum_{z_2 \in \mathbb{F}_p^*} \zeta^{-z_2} \sum_{z_3 \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q} \zeta^{Tr(az_3 x_1)} \\ &\quad \cdot \sum_{x_2 \in \mathbb{F}_q} \zeta^{Tr[(z_2 + bz_3)x_2]} \\ &= \begin{cases} p^m \sum_{z_2 \in \mathbb{F}_p^*} \zeta^{-z_2} \sum_{z_3 \in \mathbb{F}_p^*} & \\ \cdot \sum_{x_2 \in \mathbb{F}_q} \zeta^{Tr[(z_2 + bz_3)x_2]}, & a = 0, \\ 0, & a \neq 0, \end{cases} \\ &= \begin{cases} p^m \sum_{z_2 \in \mathbb{F}_p^*} \zeta^{-z_2} (p^m + \sum_{z_3 \neq -b^{-1} z_2} & \\ \cdot \sum_{x_2 \in \mathbb{F}_q} \zeta^{Tr[(z_2 + bz_3)x_2]}), & \\ & a = 0 \text{ and } b \in \mathbb{F}_p^*, \\ 0, & otherwise, \end{cases} \\ &= \begin{cases} -p^{2m}, & a = 0 \text{ and } b \in \mathbb{F}_p^*, \\ 0, & otherwise, \end{cases} \end{aligned}$$

$$
\begin{aligned}
\varphi_3 &= \sum_{x_1,x_2\in\mathbb{F}_q}\sum_{z_1\in\mathbb{F}_p^*}\zeta^{z_1(Tr(x_1^{p^s+1})-1)} \\
&\quad\cdot\sum_{z_3\in\mathbb{F}_p^*}\zeta^{z_3Tr(ax_1+bx_2)} \\
&= \sum_{z_1\in\mathbb{F}_p^*}\zeta^{-z_1}\sum_{z_3\in\mathbb{F}_p^*}\sum_{x_2\in\mathbb{F}_q}\zeta^{Tr(bz_3x_2)} \\
&\quad\cdot\sum_{x_1\in\mathbb{F}_q}\zeta^{Tr_s(2z_1x_1^{p^s+1})+Tr(az_3x_1)} \\
&= \begin{cases} -p^{m+s}\sum_{z_1\in\mathbb{F}_p^*}\zeta^{-z_1} \\ \cdot\sum_{z_3\in\mathbb{F}_p^*}\zeta^{-\frac{1}{4z_1}Tr(a^{p^s+1})z_3^2}, \\ \qquad\qquad b=0, \\ 0, \qquad\qquad b\neq 0, \end{cases} \\
&= \begin{cases} p^{m+s}(p-1), \\ \quad b=0 \text{ and } Tr(a^{p^s+1})=0, \\ (-1)^{\frac{p+1}{2}}p^{m+s+1}-p^{m+s}, \\ \quad b=0 \text{ and } Tr(a^{p^s+1})\in Sp, \\ (-1)^{\frac{p-1}{2}}p^{m+s+1}-p^{m+s}, \\ \quad b=0 \text{ and } Tr(a^{p^s+1})\in NSp, \\ 0, \qquad b\neq 0. \end{cases}
\end{aligned}
$$

Lemma 5: With the notation above, we have

I. if $a=0,b=0$, then $T=n$.

II. if $a=0,b\neq 0$,

(i) $b\in\mathbb{F}_p^*$, then $T=0$.

(ii) $b\in\mathbb{F}_q^*\backslash\mathbb{F}_p^*$, then $T=p^{2m-3}+p^{m+s-3}$.

III. if $a\neq 0,b=0$,

(i) $Tr(a^{p^s+1})=0$, then $T=p^{2m-3}+p^{m+s-2}$.

(ii) $Tr(a^{p^s+1})\in Sp$, then $T=(-1)^{\frac{p+1}{2}}p^{m+s-2}+p^{2m-3}$.

(iii) $Tr(a^{p^s+1})\in NSp$, then $T=(-1)^{\frac{p-1}{2}}p^{m+s-2}+p^{2m-3}$.

IV. if $a\neq 0,b\neq 0$

(i) $b\in\mathbb{F}_p^*$ and $Tr(a^{p^s+1})\in\{0,b^2\}$, then $T=p^{2m-3}$.

(ii) $b\in\mathbb{F}_p^*$, $Tr(a^{p^s+1})\neq 0$ and $Tr(a^{p^s+1})-b^2\in NSp$, then $T=(-1)^{\frac{p-1}{2}}p^{m+s-2}+p^{2m-3}$.

(iii) $b\in\mathbb{F}_p^*$, $Tr(a^{p^s+1})\neq 0$ and $Tr(a^{p^s+1})-b^2\in Sp$, then $T=(-1)^{\frac{p+1}{2}}p^{m+s-2}+p^{2m-3}$.

(iv) $b\in\mathbb{F}_q^*\backslash\mathbb{F}_p^*$, then $T=p^{2m-3}+p^{m+s-3}$.

Proof: The desired conclusion then follows from φ_1, φ_2, φ_3, φ_4, Lemmas 1-3 and Eq.(7). Thus we complete the proof of the lemma. □

Proof of Theorem 1: we denote the non-zero weights of the lines $1-5$ in Table I by w_i, and the corresponding multiplicity by $A_{w_i}(1\leq i\leq 5)$. Then from Lemmas 4, 5 and MacWilliams equations [8] we have

$$
\begin{aligned}
&\sum_{i=1}^{5}A_{w_i}=p^{2m}-1, \\
&\sum_{i=1}^{5}w_iA_{w_i}=p^{2m-1}(p^{2m-2}+p^{m+s+2})(p-1), \\
&\sum_{i=1}^{5}w_i^2A_{w_i}=p^{2m-2}(p-1)(p^{2m-2}+p^{m+s-2}) \\
&\cdot(p^{2m-1}+p^{m+s-1}-p^{2m-2}-p^{m+s-2}+1),
\end{aligned}
$$

where $A_1=p-1, A_2=p^m(p^m-p)$. Thus we complete the proof of Theorem 1. □

V. Concluding remarks

In this paper, we firstly constructed a class of linear codes with a few weights by a proper selection of the defining set, then we determined the parameters and the weight distributions by the character sums over finite fileds. We also wrote magma programs to verify the correctness of the results.

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$$
\begin{aligned}
\varphi_4 &= \sum_{x_1,x_2\in\mathbb{F}_q}\sum_{z_1\in\mathbb{F}_p^*}\zeta^{z_1(Tr(x_1^{p^s+1})-1)}\sum_{z_2\in\mathbb{F}_p^*}\zeta^{z_2(Tr(x_2)-1)}\sum_{z_3\in\mathbb{F}_p^*}\zeta^{z_3Tr(ax_1+bx_2)}\\
&= \sum_{z_1\in\mathbb{F}_p^*}\zeta^{-z_1}\sum_{z_2\in\mathbb{F}_p^*}\zeta^{-z_2}\sum_{z_3\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\zeta^{Tr_s(2z_1x_1^{p^s+1})+Tr(z_3ax_1)}\sum_{x_2\in\mathbb{F}_q}\zeta^{Tr((z_2+z_3b)x_2)}\\
&= \begin{cases}
-p^s\sum_{z_1\in\mathbb{F}_p^*}\zeta^{-z_1}\sum_{z_2\in\mathbb{F}_p^*}\zeta^{-z_2}(p^m\zeta^{-\frac{Tr(a^{p^s+1})}{4z_1b^2}z_2^2}+\sum_{z_3\neq -b^{-1}z_2}\zeta^{-\frac{Tr(a^{p^s+1})}{4z_1}z_3^2} & \\
\cdot\sum_{x_2\in\mathbb{F}_q}\zeta^{Tr((z_2+z_3b)x_2)}), & b\in\mathbb{F}_p^*,\\
-p^s\sum_{z_1\in\mathbb{F}_p^*}\zeta^{-z_1}\sum_{z_2\in\mathbb{F}_p^*}\zeta^{-z_2}\sum_{z_3\in\mathbb{F}_p^*}\zeta^{-\frac{Tr(a^{p^s+1})}{4z_1}z_3^2}\sum_{x_2\in\mathbb{F}_q}\zeta^{Tr((z_2+z_3b)x_2)}, & b\in\mathbb{F}_q\backslash\mathbb{F}_p^*,
\end{cases}\\
&= \begin{cases}
-p^{m+s}, & b\in\mathbb{F}_p^* \text{ and } Tr(a^{p^s+1})=0,\\
-p^{m+s}G'(\eta')\eta'(\frac{Tr(a^{p^s+1})}{4b^2})\sum_{z_1\in\mathbb{F}_p^*}\zeta^{-z_1(1-\frac{b^2}{Tr(a^{p^s+1})})}\eta'(-z_1)-p^{m+s}, & b\in\mathbb{F}_p^* \text{ and } Tr(a^{p^s+1})\neq 0,\\
0, & b\in\mathbb{F}_q\backslash\mathbb{F}_p^*,
\end{cases}\\
&= \begin{cases}
-p^{m+s}, & b\in\mathbb{F}_p^* \text{ and } Tr(a^{p^s+1})\in\{0,b^2\},\\
(-1)^{\frac{p-1}{2}}p^{m+s+1}-p^{m+s}, & b\in\mathbb{F}_p^*, Tr(a^{p^s+1})\neq 0 \text{ and } Tr(a^{p^s+1})-b^2\in NSp,\\
(-1)^{\frac{p+1}{2}}p^{m+s+1}-p^{m+s}, & b\in\mathbb{F}_p^*, Tr(a^{p^s+1})\neq 0 \text{ and } Tr(a^{p^s+1})-b^2\in Sp,\\
0, & b\in\mathbb{F}_q\backslash\mathbb{F}_p^*.
\end{cases}
\end{aligned}
$$

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