Robin's Criterion on Divisibility

Frank Vega

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Abstract Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show that the Robin inequality is true for all natural numbers $n > 5040$ that are not divisible by some prime between 2 and 1771559. We prove that the Robin inequality holds when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ for some $n > 5040$ where n' is the square free kernel of the natural number n. The possible smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$, $1 < \frac{(1+\frac{1.2762}{\log q_m}) \times \log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$ and $n < (2.82915040011)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers · Riemann zeta function

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1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real

F. Vega
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
ORCiD: 0000-0001-8210-4126
E-mail: vega.frank@gmail.com

part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [9].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 2 [4]. We extend the indivisibility property on the following result:

Theorem 1.2 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are not divisible by some prime between* 3 *and* 1771559.

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Theorem 1.3 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [4].*

In addition, we show that $\mathsf{Robins}(n)$ holds for some $n > 5040$ when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ such that n' is the square free kernel of the natural number n. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [4]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Theorem 1.4 *If n is superabundant, then n is an Hardy-Ramanujan integer [2].*

Theorem 1.5 *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

Suppose that $n > 5040$ is the possible smallest counterexample of the Robin inequality, then we prove that $q_m > e^{31.018189471}$, $1 < \frac{(1+\frac{1.2762}{\log q_m})\times\log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$ and $n < (2.82915040011)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$.

2 A Central Lemma

These are known results:

Lemma 2.1 *[4]. For $n > 1$:*

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3 *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

Proof Putting together the lemmas 2.1 and 2.2 yields the proof:

$$f(n) < \prod_{i=1}^{m} \left(\frac{q_i}{q_i-1}\right) = \prod_{i=1}^{m} \left(\frac{q_i+1}{q_i} \times \frac{1}{1-\frac{1}{q_i^2}}\right) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

3 Robin on Divisibility

We know the following lemmas:

Lemma 3.1 *[7]. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \cdots < q_m$, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i-1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Lemma 3.2 $\mathsf{Robins}(n)$ *holds for all natural numbers $10^{10^{13.11485}} \geq n > 5040$ [8].*

Theorem 3.3 *Suppose $n > 5040$. If there exists a prime $q \leq 1771559$ with $q \nmid n$, then* $\mathsf{Robins}(n)$ *holds.*

Proof We have that $f(n) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log(n)$ for any number $n > 10^{10^{13.11485}}$ since the inequality $10^{10^{13.11485}} > e^{e^{23.762143}}$ is satisfied. Note that $f(n) < \frac{n}{\varphi(n)} = \prod_{q|n} \frac{q}{q-1}$

from the lemma 2.1, where $\varphi(x)$ is the Euler's totient function. Suppose that n is not divisible by some prime $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Then,

$$\begin{aligned} f(n) &< \frac{n}{\varphi(n)} \\ &= \frac{n \times q}{\varphi(n \times q)} \times \frac{q-1}{q} \\ &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times e^{\gamma} \times \log\log(n \times q) \end{aligned}$$

and

$$\begin{aligned} \frac{f(n)}{e^{\gamma} \times \log\log(n)} &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n \times q)}{\log\log(n)} \\ &= \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n) + \log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)} \\ &= \frac{1771561}{1771560} \times \frac{q-1}{q} \times \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)}\right) \end{aligned}$$

So

$$\frac{f(n)}{e^{\gamma} \times \log\log(n)} < \frac{1771561}{1771560} \times \frac{q-1}{q} \times \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)}\right)$$

for $n \geq 10^{10^{13.11485}}$. The right hand side is less than 1 for $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Therefore, $\mathsf{Robins}(n)$ holds.

4 On the Greatest Prime Divisor

We know that

Lemma 4.1 *[6]. For $x \geq 2973$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{0.2}{\log(x)}).$$

Theorem 4.2 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $q_m > e^{31.018189471}$.*

Proof According to the theorems 1.4 and 1.5, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 3.3, we know that necessarily $q_m \geq 1771559$. So,

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{0.2}{\log(q_m)})$$

because of the lemmas 2.1 and 4.1. Hence,

$$\log\log n - \frac{0.2}{\log(q_m)} < \log q_m.$$

However, from the lemma 3.2 and theorem 3.3, we would obtain that

$$\begin{aligned} \log\log n - \frac{0.2}{\log(q_m)} &\geq 13.11485 \times \log(10) + \log\log 10 - \frac{0.2}{\log(1771559)} \\ &> 31.018189471. \end{aligned}$$

Since, we have that

$$\log q_m > \log\log n - \frac{0.2}{\log(q_m)} > 31.018189471$$

then, we would obtain that $q_m > e^{31.018189471}$ under the assumption that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

5 Some Feasible Cases

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

Lemma 5.1 $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $q \leq 7$*, where* q *is the largest prime divisor of* n*.*

Proof This is an immediate consequence of theorem 3.3.

The next theorem implies that $\mathsf{Robins}(n)$ holds for a wide range of natural numbers $n > 5040$.

Theorem 5.2 *Let* $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ *for some* $n > 5040$ *such that* n' *is the square free kernel of the natural number* n*. Then* $\mathsf{Robins}(n)$ *holds.*

Proof Let n' be the square free kernel of the natural number n, that is the product of the distinct primes $q_1, \ldots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log\log n' \leq \log\log n.$$

For all square free $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if and only if $n' \notin \{2,3,5,6,10,30\}$ [4]. However, $\mathsf{Robins}(n)$ holds for all $n > 5040$ when $n' \in \{2,3,5,6,10,15,30\}$ due to the lemma 5.1. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma} \times \log\log n'$$

because of the theorem 1.3. By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

So,

$$\begin{aligned} f(n) &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \\ &= \frac{\pi^2}{6} \times f(n') \\ &< \frac{\pi^2}{6} \times e^{\gamma} \times \log\log n' \\ &\leq e^{\gamma} \times \log\log n \end{aligned}$$

according to the formula $f(x)$ for the square free numbers [4].

6 On Possible Counterexample

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{0.2}{\log^2(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$.

Lemma 6.1 *As the prime number p_n increases, the sequence Y_n is strictly decreasing.*

Proof This lemma is obvious.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We know that

Lemma 6.2 *[10]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Lemma 6.3 *[3]. For $x \geq 2278382$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{0.2}{\log^2(x)}).$$

We will prove another important inequality:

Lemma 6.4 *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \cdots < q_m$ and $q_m > 2278382$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right).$$

Proof From the lemma 6.2, we know that

$$\theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m.$$

In this way, we can show that

$$\begin{aligned}\log\left(Y_m \times \theta(q_m)\right) &> \log\left(Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right).\end{aligned}$$

We know that

$$\begin{aligned}\log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) &= \log\left(\frac{e^{\frac{0.2}{\log^2(q_m)}}}{(1 - \frac{1}{\log(q_m)})} \times (1 - \frac{1}{\log(q_m)})\right) \\ &= \log\left(e^{\frac{0.2}{\log^2(q_m)}}\right) \\ &= \frac{0.2}{\log^2(q_m)}.\end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) \geq (\log q_m + \frac{0.2}{\log^2(q_m)}).$$

Due to the lemma 6.3, we prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (\log q_m + \frac{0.2}{\log^2(q_m)}) < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right)$$

when $q_m > 2278382$.

We use the following lemma:

Lemma 6.5 *[7]. Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i + 1}}\right).$$

The following theorems have a great significance, because these mean that the possible smallest counterexample of the Robin inequality greater than 5040 must be very close to its square free kernel.

Theorem 6.6 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $(\log n)^{\beta} < Y_m \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

Proof According to the theorems 1.4 and 1.5, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 4.2, we know that necessarily $q_m > e^{31.018189471}$. From the lemma 6.5, we note that

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

However, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right)$$

because of the lemma 6.4 when $q_m > 2278382$. If we multiply by $\prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$ the both sides of the previous inequality, then we obtain that

$$f(n) < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

If n is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^{\gamma} \times \log\log n < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

because of

$$e^{\gamma} \times \log\log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \times \log\log n < \log\left(Y_m \times \log(N_m)\right)$$

which is equivalent to

$$(\log n)^{\beta} < Y_m \times \log(N_m)$$

where $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$. Therefore, the proof is done.

Theorem 6.7 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1,\ldots,a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

Proof From the theorem 4.2, we know that necessarily $q_m > e^{31.018189471}$. Using the theorem 6.6, we obtain that

$$(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$$

due to the lemma 6.1 since we have that $Y_m < 1.03352795481$ when $q_m > e^{31.018189471}$.

Theorem 6.8 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1,\ldots,a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $n < (2.82915040011)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.*

Proof According to the theorems 1.4 and 1.5, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the lemma 6.4, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right) = e^{\gamma} \times \log\log(N_m^{Y_m})$$

for $q_m > 2278382$. In this way, if $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $n < N_m^{Y_m}$ since by the lemma 2.1 we have that

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

That is the same as $n < N_m^{Y_m - 1} \times N_m$. We can check that $q_m^{Y_m - 1}$ is monotonically decreasing for all primes $q_m > e^{31.018189471}$. Certainly, the derivative of the function

$$g(x) = x^{\left(\frac{e^{\frac{0.2}{\log^2(x)}}}{(1 - \frac{1}{\log(x)})} - 1\right)}$$

is less than zero for all real numbers $x \geq e^{31.018189471}$. Consequently, we would have that

$$q_m^{Y_m - 1} < g(e^{31.018189471}) < 2.82915040011$$

for all primes $q_m > e^{31.018189471}$. Moreover, we would obtain that

$$q_m^{Y_m - 1} > q_j^{Y_m - 1}$$

for every integer $1 \leq j < m$. Finally, we can state that $n < (2.82915040011)^m \times N_m$ since $N_m^{Y_m - 1} < (2.82915040011)^m$ when $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

We know the following results:

Lemma 6.9 *[5]. For $x > 1$:*

$$\pi(x) \leq (1 + \frac{1.2762}{\log x}) \times \frac{x}{\log x}$$

where $\pi(x)$ is the prime counting function.

Lemma 6.10 *If $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $p < \log n$ where p is the largest prime divisor of n [4].*

Theorem 6.11 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $1 < \frac{(1+\frac{1.2762}{\log q_m}) \times \log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.*

Proof Note that $n < (2.82915040011)^m \times N_m$ when n is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. If we apply the logarithm to the both sides, then

$$\log n < m \times \log(2.82915040011) + \log N_m.$$

According to the lemma 6.9, we have that

$$\log n < (1 + \frac{1.2762}{\log q_m}) \times \frac{q_m}{\log q_m} \times \log(2.82915040011) + \log N_m.$$

From the lemma 6.10, we would have

$$\log n < (1 + \frac{1.2762}{\log q_m}) \times \frac{\log n}{\log\log n} \times \log(2.82915040011) + \log N_m.$$

which is the same as

$$1 < \frac{(1 + \frac{1.2762}{\log q_m}) \times \log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$$

after of dividing by $\log n$.

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